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Difference Equations and Conservation Laws

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1. INTRODUCTION

In the fall of 1951 I joined the Institute for Advanced Study in Princeton. In physics, that was a time of quiet following the burst of activity in renormalization theory, and before the strange particle physics, which was to come a couple of years later. The younger physics members at the Institute decided to have a special seminar on statistical mechanics, to the dismay of Oppenheimer, who felt that somehow this would not be the best use of our energy, nor in the best interests of the Institute. The seminar consisted of a series of lectures by Jan de Boer and H. A. Kramers. Both gave very good lectures. Then one day a new speaker came whom I had not heard before. His talk was even better; it was exceptionally clear and extremely lively. I thought, here is a physicist who not only knows his stuff, but can truly communicate with his audience. That was the first time I met Mark Kac.

This began a friendship that lasted more than three decades. My opinion of Mark Kac as a physicist remained the same, with the added knowledge that he also had an equally exceptional ability as a pure mathematician. To me, Mark was a physicist, but did mathematics in his spare time.

As is well known, Mark made fundamental contributions to statistical mechanics, the Feynman formulation of quantum mechanics, and the soliton solutions in field theory. Throughout the years I learned a great deal of both physics and mathematics from him. Discussions with him had a cleansing effect on one's mind. The most recent time, before Mark moved

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to the West Coast, was the lectures he and Ken Case gave me on solitons, which inspired my work on the nontopological species.

The subject I discuss in this paper is difference equations and conservation laws. This topic is related to my recent work, which, unfortunately, could not have the benefit of Mark's advice and criticism. Nevertheless, I feel, were he here, he might like the basic ideas and would see how the whole thing could be improved. It is to the memory of my dear friend, Mark Kac, that I dedicate this paper.

At present, in almost all branches of physics, fundamental laws are always expressed in terms of differential equations. Difference equations are usually used only as approximations. In this paper I wish to explore an alternative point of view: that physics should be formulated in terms of difference equations and that these difference equations could exhibit all the desirable symmetry properties and conservation laws. As we shall see, the particular class of difference equations that will be discussed contains more information and more symmetry than the corresponding differential equations.

2. TIME AS A DYNAMICAL VARIABLE

In this new theory I shall treat time as a dynamical variable.[1] This will lead to a dynamics formulated in terms of difference equations, instead of the usual differential equations. I first review briefly the classical theory of this new mechanics, called discrete mechanics, and then go over to the quantum theory.

2.1. Classical Mechanics

Take the simplest example of a nondimensional nonrelativistic particle of unit mass moving in a potential $V(x)$. In the usual continuum mechanics the action is

$$A(x(t))=\int_0^T \left[\tfrac{1}{2}\dot{x}^2 - V(x)\right] dt \tag{2.1}$$

where $x(t)$ can be any smooth function of the time t. Keeping fixed the initial and final positions, say x_0 and x_f, at $t=0$ and T, we determine the orbit of the particle by the stationary condition

$$\delta A/\delta(x(t))=0 \tag{2.2}$$

which leads to Newton's equation

$$\ddot{x} = -dV/dx \tag{2.3}$$

In the above, x is the dynamical variable and t is merely a parameter. Next, we shall see how this customary approach may be modified in the discrete version.

Let the initial and final positions of the particle be the same,

$$x_0 \quad \text{at} \quad t=0, \qquad x_f \quad \text{at} \quad t=T \tag{2.4}$$

In the discrete mechanics we restrict the usual smooth path $x(t)$ to a "discrete path" $x_D(t)$, which is continuous but piecewise linear, characterized by N vertices (as shown in Fig. 1). In Fig. 1a we have the usual smooth path $x(t)$ of a nonrelativistic particle in classical mechanics. Moving along $x(t)$ from $t=0$ to $T>0$, the time t increases monotonically; this property is retained under the constraint restricting $x(t)$ to $x_D(t)$. Thus, as in Fig. 1b, we may label the N vertices of $x_D(t)$ consecutively as $n=1$, 2,..., N, each of which carries a space-time position x_n and t_n with

$$0<t_1<t_2<t_3<\cdots<t_N<T \tag{2.5}$$

The nearest neighboring vertices are linked by straight lines, forming the discrete path $x_D(t)$, which also appears as a one-dimensional lattice with n as lattice sites. In Fig. 1b, a variation of the space-time positions of these vertices changes the discrete path $x_D(t)$. However, a mere exchange of any two vertices clearly defines the same $x_D(t)$. This is because only the discrete path with unlabeled vertices has a physical meaning. There is no

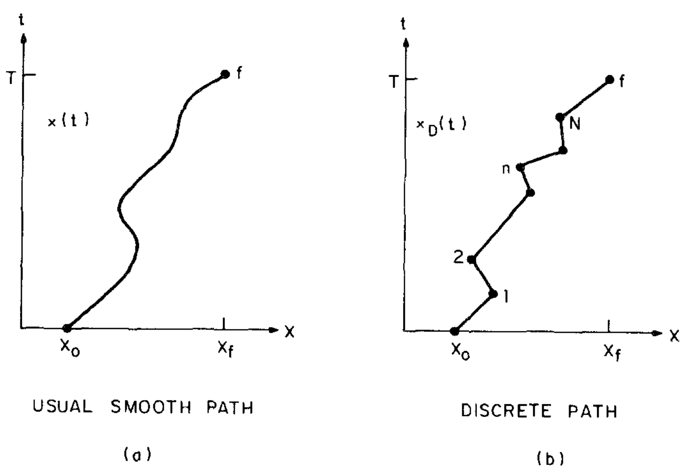


Fig. 1. Examples of a smooth path (a) and a discrete path (b) from $x=x_0$ at $t=0$ to $x=x_f$ at $t=T$.

"individual" identity of any of the vertices. Thus, the time-ordered sequence (2.5) is not an additional restriction, but one that arises naturally when we pass from the usual $x(t)$ to the discrete $x_D(t)$.

In the following, we shall keep the site density

$$N/T \equiv 1/l \tag{2.6}$$

fixed, and regard l as a fundamental constant of the theory. The action integral (2.1) evaluated on such a discrete path $x_D(t)$ is

$$A_D = A(x_D(t)) = \sum_n \left[\frac{1}{2} \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} - (t_n - t_{n-1}) \bar{V}(n) \right] \tag{2.7}$$

where

$$\bar{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x)\, dx \tag{2.8}$$

is the average of $V(x)$ along the straight line between x_{n-1} and x_n.

Because the path $x_D(t)$ is completely specified by its vertices $n(x_n, t_n)$, a variation in $x_D(t)$ is equivalent to a variation in all the positions of its vertices

$$d[x_D(t)] = \prod_n [dx_n][dt_n] \tag{2.9}$$

Correspondingly, the dynamical equation (2.2) becomes the difference equations

$$\partial A_D / \partial x_n = 0 \tag{2.10}$$

$$\partial A_D / \partial t_n = 0 \tag{2.11}$$

We see that in this new mechanics the roles of x_n and t_n are quite similar. Both appear as *dynamical variables.* For each x_n or t_n we have one difference equation, (2.10) or (2.11). The former gives Newton's law on the lattice and the latter gives the conservation of energy

$$E_n \equiv \frac{1}{2} \left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}} \right)^2 + \bar{V}(n) = E_{n+1} \tag{2.11$'$}$$

In the usual continuum mechanics, conservation of energy is a consequence of Newton's equation. Here, these two equations (2.10) and (2.11) are independent. Altogether there are $2N$ such equations, matching in number the $2N$ unknowns x_n and t_n in the problem. Because the action A_D is stationary under a variation in x_n and in t_n for every n, the discrete theory retains the translational invariance of both space and time, and that leaves the conservation laws of energy and momentum intact.[2]

For a free particle $V(x)=0$, (2.11) and (2.10) become degenerate; both give

$$v_n=\frac{x_n-x_{n-1}}{t_n-t_{n-1}}=\text{const}$$

The corresponding trajectory is a straight line, the same as the continuum case.

When $V(X)=gx$ with g a constant, the solution of (2.10) and (2.11) can be readily found. We find in this case the spacing between successive t_n to be independent of n:

$$t_n-t_{n-1}=\varepsilon=\text{const}$$

Correspondingly, $t_n=t_0+n\varepsilon$ and

$$x_n=x_0+nv_1\varepsilon-\tfrac{1}{2}n(n-1)\,g\varepsilon^2$$

where v_1 is the initial velocity $(x_1-x_0)/(t_1-t_0)$.

When $l\to 0$, the site density approaches ∞ and the discrete path $x_D(t)$ can assume the form of any smooth path $x(t)$; consequently, the discrete mechanics approaches the usual continuum mechanics. Introduce

$$\tau\equiv nl$$

which varies from 0 to T as n runs from 0 to N. Regard

$$x_n=x(\tau),\qquad t_n=t(\tau)$$

From (2.10) and (2.11), it can be shown that in the limit $l\to 0$, but keeping T fixed (hence, $N\to\infty$),

$$\frac{d^2x}{dt^2}=-\frac{dV}{dx}\tag{2.12}$$

and

$$\left(\frac{dt}{d\tau}\right)^3\left(\frac{dV}{dx}\right)^2=\text{const}\tag{2.13}$$

[2] Here, conservation of momentum means that the change of particle momentum is equal to the "impulse" generated by the potential.

The former is Newton's equation, and the latter gives the asymptotic distribution of t_n versus n. The constant in (2.13) is determined by the boundary condition (2.4), so that when τ varies from 0 to T, t also changes from 0 to T. In the usual continuum mechanics, only (2.12) is retained. Therefore, even in this limit, the discrete mechanics contains more information than the usual continuum mechanics. From (2.13), we see that, except for $V(x) = gx$, the spacing $t_n - t_{n-1}$ is not a constant.

It is of interest to examine the distribution $t(\tau)$ near the point $V'(x) \equiv dV/dx = 0$, which occurs at, say, $x = \bar{x}$. Let the particle trajectory in the continuum limit be $x = x(t)$. When $x = \bar{x}$, we have $V'(\bar{x}) = 0$ and, for the solution under consideration, $t = \bar{t}$, so that $\bar{x} = x(\bar{t})$. In the neighborhood x near $\bar{x}$, we may write, with $V''(x) \equiv d^2V/dx^2$ and $\dot{x} \equiv dx/dt$,

$$\begin{aligned} V'(x) &\approx (x - \bar{x})\, V''(\bar{x}) \\ &= (t - \bar{t})\, \dot{x}(\bar{t})\, V''(\bar{x}) \end{aligned}$$

Substituting this expression into (2.13), we find

$$(t - \bar{t}) \propto (\tau - \bar{\tau})^{3/5}$$

Hence, as $\tau \to \bar{\tau}$ (correspondingly, $n \to \bar{\tau}/l$), although $dt/d\tau \to \infty$, one sees that $t \to \bar{t}$ and remains finite. Information such as this is lost if one concentrates only on Newton's equation (2.12).

In the following, we are interested in $l \neq 0$, in which case the discrete mechanics is fundamentally different from the continuum theory.

2.2. Nonrelativistic Quantum Mechanics

When we go over from classical to quantum mechanics, in the usual continuum theory the particle can take on any smooth path $x(t)$; each path carries an amplitude e^{iA}, where $A = A(x(t))$ is the same action integral (2.1). In Feynman's path integration formalism, the matrix element of e^{-iHT} in the usual continuum quantum mechanics is given by

$$\langle x_f | \, e^{-iHT} \, | x_0 \rangle = \int e^{iA(x(t))} d[x(t)]$$

in which all paths $x(t)$ have the same endpoints (2.4) and

$$H = -\tfrac{1}{2}\, \partial^2/\partial x^2 + V(x)$$

Sometimes it is more convenient to consider the analytic continuation of T

to $-iT$. The operator e^{-iHT} becomes then e^{-HT}, and its matrix element is given by

$$\langle x_f | \, e^{-HT} \, | x_0 \rangle = \int e^{-\mathscr{A}(x(t))} d[x(t)] \tag{2.14}$$

where

$$\mathscr{A}(x(t)) = \int_0^T \left[\tfrac{1}{2}\dot{x}^2 + V(x)\right] dt \tag{2.15}$$

In the corresponding discrete theory, we again restrict the particle to move only along the discrete path $x_D(t)$. By using (2.7) and (2.9), we see that (2.12) becomes

$$\int e^{iA_D} \prod_n \, [dx_n][dt_n] \tag{2.16}$$

Likewise, (2.14) and (2.15) become

$$\langle x_f | \, G_N(T) \, | x_0 \rangle \equiv \int e^{-\mathscr{A}_D} \prod_{n=1}^{N} \, [dx_n][dt_n] \tag{2.17}$$

where

$$\mathscr{A}_D = \mathscr{A}(x_D(t)) \tag{2.18}$$

When the vertices $n = 1, 2, ...$ are arranged in a time-ordered sequence (2.5), by using (2.15) and (2.18) we see that the discrete action $\mathscr{A}_D$ is given by

$$\mathscr{A}_D = \sum_{n=1}^{N+1} \left[\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})} + (t_n - t_{n-1}) \, \bar{V}(n) \right] \tag{2.19}$$

with $x_{N+1} = x_f$ and $t_{N+1} = T$, as shown in Fig. 2b.

In the integration over $\prod_n [dt_n]$, whenever t_i appears larger than, say, t_{i+1}, we should relink the vertices so that the newly linked ones are in a time-ordered sequence. Alternatively, we may relabel them so that (2.5) remains valid; such a relabeling of vertices clearly does not change the discrete path $x_D(t)$. [As explained before, this follows from the usual nonrelativistic continuum mechanics in which the path $x(t)$ is a single-valued function of t.]

In the quantum version of the discrete mechanics it is more convenient to regard the constraint (2.6) as a condition on the *average* site density. This can be most easily arranged by considering an ensemble sum over N:

$$\mathscr{G}(T, l) \equiv \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{1}{l}\right)^N G_N(T) \tag{2.20}$$

where $G_N(T)$ refers to the matrix defined by (2.17). One may readily verify that this Green's function satisfies

$$\frac{\partial}{\partial(1/l)} \mathscr{G}(T, l) = \int_0^T \mathscr{G}(\tau, l)\, \mathscr{G}(T-\tau, l)\, d\tau \tag{2.21}$$

from which it follows that for large T and neglecting $e^{-T/l}$, the operator $\mathscr{G}(T, l)$ becomes

$$\mathscr{G}(T, l) \sim e^{-\mathscr{H}T} \tag{2.22}$$

where $\mathscr{H}$ is Hermitian. When $l \to 0$, $\mathscr{H}$ reduces to the continuum Hamiltonian H, given by (2.13). The analytic continuation of $\mathscr{G}(T, l)$ from T to iT leads at large T to the unitary operator $e^{-i\mathscr{H}T}$, which is the S-matrix of the theory. Therefore, the unitarity of the S-matrix is maintained in the new mechanics,[2] at least when $\mathscr{H}$ is $O(1)$.

3. RELATIVISTIC QUANTUM FIELD THEORY

As an example, let $\phi(x)$ be a scalar field in the usual continuum theory, with x denoting the space-time coordinates. In the path integration formulation the operator e^{-HT} is given by, similar to (2.14),

$$e^{-HT} = \int e^{-\mathscr{A}}[d\phi(x)] \tag{3.1}$$

where H is the Hamiltonian operator, $\mathscr{A}$ the usual continuum action in the Euclidean space, and T the total "time" interval. [Here, as in (2.14)–(2.15), "time" refers to the Euclidean time.] Because in the usual continuum theory the space-time coordinates x are parameters, and only $\phi(x)$ are dynamical variables, the functional integration in (3.1) is over $[d\phi(x)]$, not $[dx]$.

In the discrete version, we impose a constraint on the (average) number N of experiments that can be performed within any given space-time volume Ω, with $N/\Omega \equiv l^{-4} =$ fundamental constant. Each measurement determines the field $\phi(i)$ as well as the space-time position $x(i)$ with $i = 1, 2, ..., N$. The i will be referred to as lattice sites, as illustrated by Fig. 2a.

As we shall see, the Green's function (3.1) will be replaced by

$$\int e^{-\mathscr{A}_D}[dx(i)][d\phi(i)] \tag{3.2}$$

Because $\phi(i)$ and $x(i)$ are all dynamical variables, in the discrete theory we integrate over $[d\phi(i)]$ as well as $[dx(i)]$. The latter integration makes it

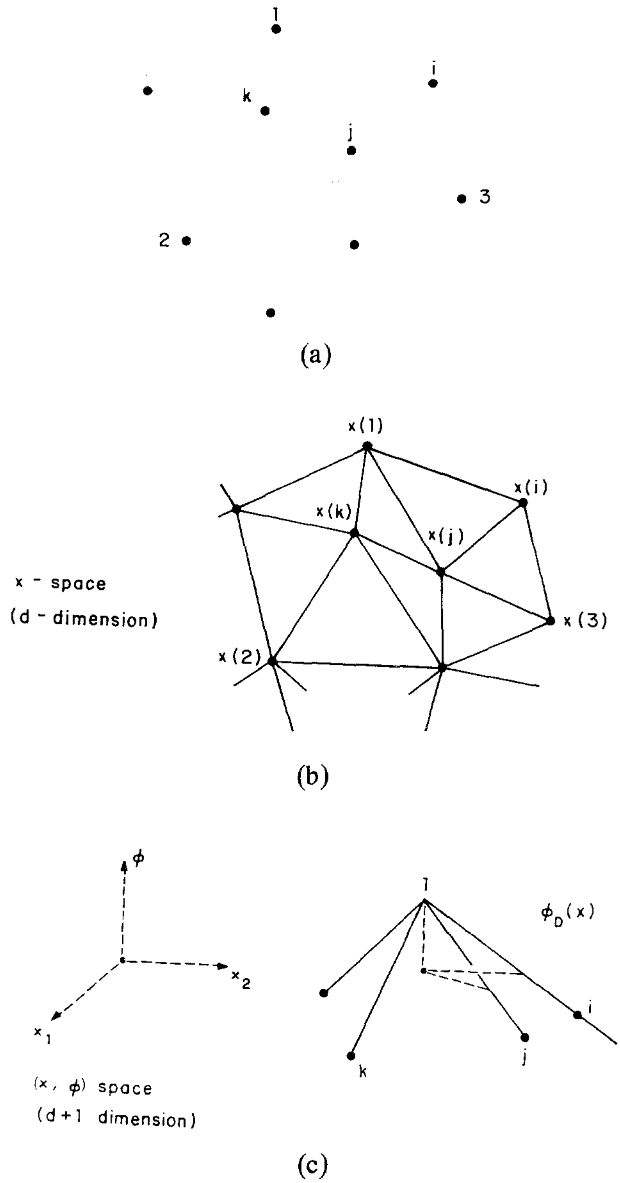


Fig. 2. (a) N measurements in a volume Ω in d-dimension, each measurement represented by a point of coordinates $x_1, x_2, \ldots, x_d$. (b) N points connected into a d-dimensional simplicial lattice. (c) Because each point also carries the field strength ϕ that is being measured, the d-dimensional simplicial lattice is embedded in a $d+1$ dimensional space with coordinates ϕ, $x_1, \ldots, x_d$.

obvious that rotational and translational symmetries can be maintained in the discrete theory.

To simulate the local character of the usual continuum theory, each site in the discrete theory is coupled only to its neighboring sites, as illustrated in Fig. 2b. The whole volume is then divided into triangles if the dimension of $x(i)$ is $d=2$, tetrahedra if $d=3$, 4-simplices when $d=4$, etc. An example of such a simplicial lattice when $d=2$ appears in Fig. 2b.

We give the algorithm[3] of linking an arbitrary distribution of sites into a simplicial lattice for any dimension d: select any group of $d+1$ sites. Consider the hypersphere (in the d-dimensional Euclidean space) whose surface passes through these $d+1$ sites. If the interior of the sphere is empty of sites, link these sites to form a d-simplex; otherwise, do nothing. Proceed to another group of $d+1$ sites, and repeat the same steps. The d-simplices thus formed never intersect each other, and the sum total of their volumes fills the entire space.

Each site i carries, in addition to its space-time coordinates $x(i)$, also a $\phi(i)$. Viewed in the x–ϕ space, the lattice forms a d-dimensional surface represented by $\phi_D(x)$, called the "discrete" function; it is continuous but piecewise flat within each d-simplex, as illustrated in Fig. 2c.

The discrete action $\mathscr{A}_D$ in (3.2) can be readily evaluated by using the usual continuum action $\mathscr{A}(\phi(x))$, but restricting $\phi(x)$ to the discrete function

$$\mathscr{A}_D \equiv \mathscr{A}(\phi_D(x)) \tag{3.3}$$

For example, if

$$\mathscr{A}(\phi(x)) = \int [\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)]\, dx \tag{3.4}$$

where dx is the d-dimensional volume element in the x space, then, setting $\phi(x)$ to be the discrete function $\phi_D(x)$, we find

$$\mathscr{A}_D = \mathscr{A}(\phi_D(x)) = \tfrac{1}{2}\sum_{l_{ij}} \lambda_{ij}([\phi(i)-\phi(j)]^2) + \sum_i \omega_i V(\phi(i)) \tag{3.5}$$

where the first sum is over all links l_{ij} and the second over all sites i, ω_i is the volume of the Voronoi cell that is dual to the site i, and[4]

$$\lambda_{ij} = -\frac{1}{d^2}\sum \frac{1}{V(ij)}\,\tau(i)\cdot\tau(j) \tag{3.6}$$

in which the sum extends over all d-simplices $V(ij)$ that share the link l_{ij}. In $V(ij)$, each vertex, say k, faces a $(d-1)$-dimensional simplex $\tau(k)$. In (3.6), $V(ij)$ also denotes the volume of the d-simplex and $\tau(i)$ is the outward normal vector of $\tau(i)$ times its $(d-1)$-dimensional volume, as illustrated in Fig. 3.

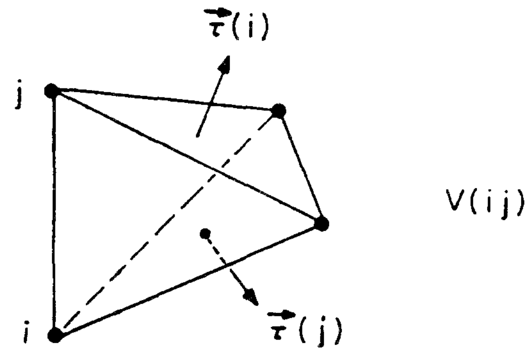


Fig. 3. $V(ij)$ represents a d-simplex that shares the link l_{ij}. Let $\tau(i)$ be its $d-1$-dimensional simplex (on its surface) that faces the vertex i, and $\tau(i)$ be the outward normal vector of $\tau(i)$ times the $d-1$ dimensional volume of $\tau(i)$.

As in the previous section, mathematically the discrete theory can be regarded as a special case of the usual continuum theory, one in which $\phi(x)$ is restricted to those continuous but piecewise flat functions $\phi_D(x)$ with a fixed average density of vertices (i.e., lattice sites). Because the site density is an invariant, rotational and translational invariances can both be preserved in the discrete theory.

Since the discrete surface, described by $\phi_D(x)$, is characterized by the positions $\phi(i)$ and $x(i)$ of its vertices, a variation over the functional space $\phi(x)$ in the usual continuum theory becomes

$$[d\phi_D(x)] = \prod_i [d\phi(i)][dx(i)] \tag{3.7}$$

Correspondingly, (3.1) becomes (3.2). As $x(i)$ changes, the linking algorithm keeps track of how these vertices should be linked, so that the discrete action $\mathscr{A}_D$ is extensive; i.e., $\mathscr{A}_D$ is proportional to the overall space-time volume Ω when Ω is large. Thereby, the unitarity of the S-matrix can be established, as before.

In the usual continuum theory, the equation of motion is given by the partial differential equation

$$\delta\mathscr{A}(\phi(x))/\delta\phi(x) = 0 \tag{3.8}$$

Here in the discrete version it is replaced by the difference equations

$$\partial\mathscr{A}_D/\partial\phi(i) = 0, \qquad \partial\mathscr{A}_D/\partial x(i) = 0 \tag{3.9}$$

The former is the field equation on the lattice and the latter expresses the conservation law of the energy-momentum tensor.

In the integrand of (3.2), the locations of x can be arbitrary. Hence, the discrete action $\mathscr{A}_D$ is identical to that of a random lattice.[5]

4. GAUGE THEORY

I review briefly the random lattice results on Abelian (QED) and non-Abelian (QCD) gauge theories.

The lattice gauge theory was introduced by K. Wilson. In the strong coupling limit (square of coupling constant $g^2 \to \infty$), any lattice gauge theory gives confinement. This holds for both QED and QCD, and for arbitrary space dimension d. The realistic case corresponds, however, to the weak coupling. Thus, a key question is whether the transition from strong to weak coupling is smooth or not. If smooth, then the confinement property of the strong coupling can be carried over to weak coupling; otherwise, it cannot. When $\beta = 1/g^2$ changes from 0 (strong coupling) to ∞ (weak coupling), we would like the transition to be smooth for the non-Abelian case, but not smooth for the Abelian case, so that the confinement holds for QCD, but no for QED. In a hypercubic lattice, there appears to

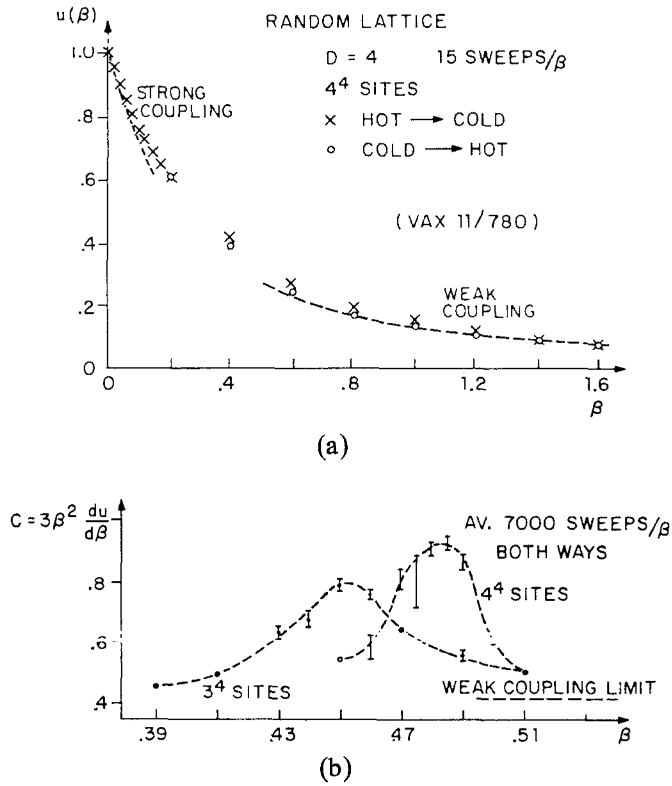
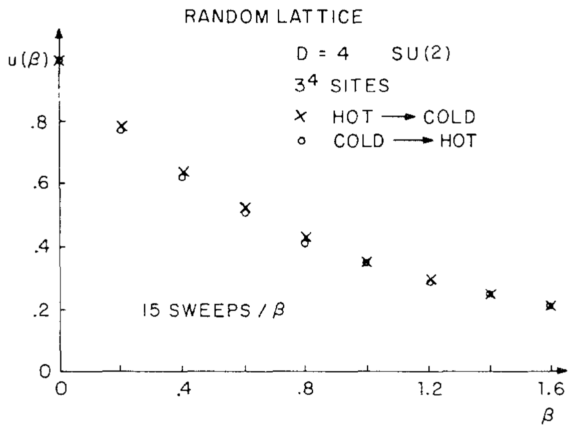


Fig. 4. The energy $u(\beta)$ per plaquette (a) and the specific heat (b) vs. $\beta \equiv 1/g^2$ for the $U(1)$ theory on a random $4 \times 4 \times 4 \times 4$ lattice.

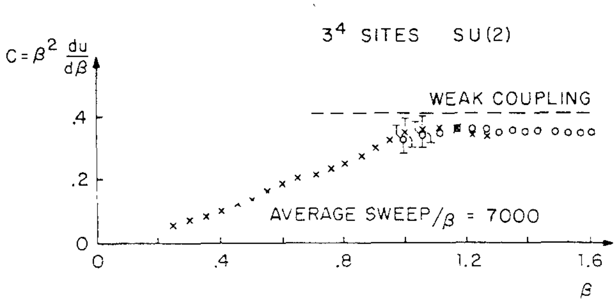
be a phase transition in β for the $U(1)$ gauge, consistent with the fact that QED is not confined. However, numerical results in $SU(2)$ and $SU(3)$ indicate that the transition from $\beta=0$ to $\beta=\infty$ is also far from smooth. While there is probably no bona fide phase transition in the non-Abelian case, the change from cubic (when $g^2=\infty$) to spherical (when $g^2=0$) symmetry is sufficiently hazardous that it is difficult to infer, from the strong coupling result, that confinement would remain valid in the weak coupling case.

On the other hand, for the random lattice, its strong coupling limit behaves like a relativistic string theory, with full rotational symmetry: the string thickness t is related to the string tension T by

$$t^2 = (1/2\pi T) \ln a \tag{4.1}$$



(a)



(b)

Fig. 5. The energy $u(\beta)$ per plaquette (a) and the specific heat (b) vs. $\beta \equiv 1/g^2$ for the $SU(2)$ theory on a random $3\times3\times3\times3$ lattice.

where a is the area enclosed by the string. Furthermore, the mass of the glueball m_J for a large angular momentum J varies as

$$m_J \propto \sqrt{J} \tag{4.2}$$

exhibiting the typical Regge behavior of a rotating relativistic string. Both (4.1) and (4.2) are valid in the strong coupling limit.

Numerical programs for a random lattice gauge theory were set up by Friedberg and Ren at Columbia; the computations were carried out by Ren.[6] Figures 4a and 4b give the average plaquette energy u and specific heat C versus $\beta = 1/g^2$ for the $U(1)$ theory.

The corresponding plots for an $SU(2)$ theory are given in Figs. 5a and 5b. We see that the specific heat has a peak in the $U(1)$ theory, but not in the $SU(2)$ theory. For $U(1)$, the peak becomes steeper when the number of lattice sites increases, suggesting that there is a phase transition. On the other hand, the specific heat curve for $SU(2)$ has no peak, indicating that the passage from strong to weak coupling is a smooth one. Consequently, while both theories are confined in the strong coupling limit, the weak coupling limit is consistent with deconfinement in the $U(1)$ theory (QED), but with confinement in a non-Abelian gauge theory (QCD).

In contrast, Fig. 6 gives the numerical calculation by Christ and Terrano[7] for the $SU(3)$ gauge theory on a regular lattice. As we can see,

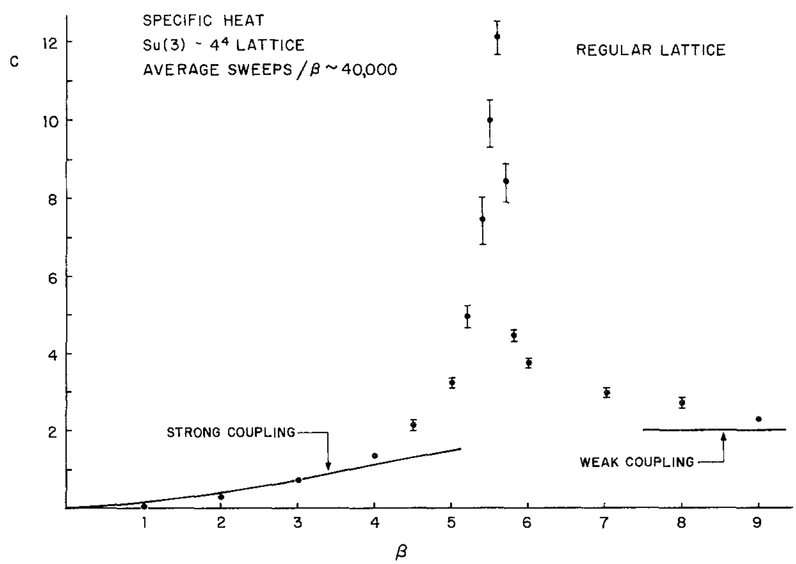


Fig. 6. The specific heat versus β for the $SU(3)$ theory on a regular $4 \times 4 \times 4 \times 4$ lattice.

there is a sharp peak in the specific heat, suggesting that the transition from strong to weak in a regular lattice is by no means smooth, unlike that in a random lattice.

5. LATTICE GRAVITY

The usual Einstein action in general relativity is

$$A(S) = \int_S |g|^{1/2} \mathscr{R}\, dx \tag{5.1}$$

where S is a d-dimensional, smooth, continuous surface, $|g|$ is the absolute value of the determinant of the matrix of the metric tensor $g_{\mu\nu}$ on S, $\mathscr{R}$ is the scalar curvature, and dx is the d-dimensional volume element in the space-time coordinate x.

For lattice gravity, we consider first a (random) lattice $\mathring{L}$ in a flat d-dimensional Euclidean space R_d. Label each site by $i = 1, 2,...$. For every linked pair of sites i and j there is a link length $\mathring{l}_{ij}$.

Consider now an arbitrary variation

$$\mathring{l}_{ij} \to l_{ij} \tag{5.2}$$

Correspondingly, each d-simplex, say $\mathring{\tau}$ in $\mathring{L}$, becomes a new d-simplex τ with the same vertices, but different link lengths. These new link lengths l_{ij} are assumed to satisfy all simplicial inequalities, so that each d-simplex τ, by itself, can still be realized in a flat d-dimensional space R_d. In general the entire new lattice cannot fit into R_d. This then defines[3] a d-dimensional nonflat lattice surface L.

Sometimes, it is convenient to embed L in a flat space R_N. This is possible if

$$N = d + n$$

is sufficiently large; in that case

$$l_{ij}^2 = [\mathbf{r}(i) - \mathbf{r}(j)]^2 \qquad \text{in} \quad R_N \tag{5.3}$$

with $\mathbf{r}(i)$ the Cartesian N-dimensional position vector of the ith site in R_N. Since, as we shall see, we shall deal only with the intrinsic geometric properties of the lattice surface, this embedding is merely a convenience.

[3] For physical application to general relativity, in order to maintain the quasilocal character of the discrete action, we must link only neighboring sites. Thus, when the new link lengths l_{ij} are too large, the sites have to be relinked. Details will be given elsewhere.

Next we wish to evaluate the Einstein action (5.1) when S is restricted to the lattice surface L. At first sight, it might appear difficult because the metric g_{ij} would change discontinuously from simplex to simplex, the Christoffel symbol would then acquire δ-functions, and the scalar curvature δ'-functions. Since the Einstein action is nonlinear in g_{ij}, one might expect the resulting expression to be totally unmanageable. It turns out that this is not so.

It can be shown that the Einstein formula (5.1) evaluated on any d-dimensional lattice space L gives the discrete action[8,9]

$$A(L) \equiv \int_L |g|^{1/2} \mathscr{R}\, dx \tag{5.4}$$

$$= 2 \sum_s s\varepsilon_s \tag{5.5}$$

where dx is the d-dimensional volume element, s is the volume of the $(D-2)$-simplex, ε_s is Regge's deficit angle around s, and the sum extends over all s in the lattice (see Ref. 10 for the definition of ε_s). The right-hand side of (5.5) is *precisely* the formula of Regge calculus.[10]

In Regge's original approach, he considered the discrete action as an approximation to Einstein's continuum action. Here we are reversing the role and regarding the discrete action $A(L)$ as more fundamental. It is therefore satisfying to realize that Regge's action is identical to Einstein's action, but evaluated on L.

The quintessence of Einstein's theory of general relativity lies in its invariance under a general coordinate transformation

$$x \to x' \tag{5.6}$$

that leaves ds^2 unchanged. Since the action for the lattice space L is the discrete action

$$A_D \equiv A(L) = \int_L |g|^{1/2} \mathscr{R}\, dx = 2 \sum_s s\varepsilon_s \tag{5.7}$$

the discrete theory clearly remains invariant under the coordinate transformation (5.6). Thus, the entire apparatus of coordinate invariance in the usual continuum theory automatically applies to the lattice theory as well. In addition, as we shall see, the lattice theory enjoys still another, totally new class of symmetries, which does not exist in the usual continuum theory. Aesthetically, this adds greatly to the appeal of lattice gravity. For physical applications, when the link length l is small, our general formula (5.7) ensures that all known tests of general relativity are automatically satisfied. Furthermore, by keeping l nonzero, we see that the lattice action A_D per volume possesses only a finite degree of freedom. The normal difficulty of ultraviolet divergence that one encounters in quantum gravity disappears in the lattice theory. All these suggest that the lattice theory with a nonzero l may be more fundamental. The usual continuum theory is quite possibly only an approximation.

To amplify the aforementioned symmetry properties, let us consider any lattice L. From (5.7), we see that the discrete action A_D, through its right-hand side, is a function of the link lengths l_{ij}:

$$A_D = A_D(l_{ij}) \tag{5.8}$$

We may also characterize the lattice by other means of parametrization. We assume all the lattice sites i to lie on a d-dimensional smooth enveloping surface S, with $z_\mu(i)$ as the coordinates of the site i on S, where $\mu = 1, 2,..., d$. Embed both S and L in a flat space R_N, which is always possible provided that N is sufficiently large. Because of (5.3), l_{ij} can also be determined by giving S and $z_\mu(i)$. Hence, we can also express A_D as a function of the enveloping surface S and the site positions on S:

$$A_D = A_D(S, z_\mu(i)) \tag{5.9}$$

Thus, we can have new symmetry transformations:

(i) Fix $z_\mu(i)$, but vary $S \to S'$.

(ii) Fix S, but vary $z_\mu(i) \to z'_\mu(i)$.

These symmetries are exact if the l_{ij} are unchanged; they can be approximate even if the l_{ij} do change, provided that, e.g., the link lengths are sufficiently small and $|g|^{1/2} \mathscr{R}\, dx$ remains the same on the enveloping surface, in which case $A_D \approx A(S)$ of (5.1).

In the usual continuum theory, the physical space-time points and the underlying four-dimensional manifold are the same. Here, they are distinct; the former is related to measurements, while the latter is purely a mathematical artifice (like the choice of gauge in the usual continuum theory of a spin-1 or -2 field).

6. CONCLUDING REMARKS

For more than three centuries we have been influenced by the precept that fundamental laws of physics should be expressed in terms of differential equations. Difference equations are always regarded as approximations. Here, I try to explore the opposite: Difference equations are more fundamental, and differential equations are regarded as approximations.

As I have shown, such a difference equation formulation leads to the discrete mechanics, which can also be viewed as the mathematical limit of the usual continuum mechanics, but with a fixed density of lattice sites. Because this is an invariant constraint, the discrete theory shares the symmetries of the usual continuum theory. In this way, I have succeeded in the creation of theories with finite degrees of freedom, but which retain all the good properties of the usual continuum theory. I suggest that this discrete formulation might be more fundamental.

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